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# Planform selection in salt fingers

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The problem of small-aspect-ratio thermohaline convection is discussed in conditions appropriate to the salt-finger regime. Two-scale methods are employed to produce nonlinear coupled evolution equations for an arbitrary number of interacting roll solutions. These are solved in simple cases and it is shown that roll-type planforms are preferred over square cells throughout the range of validity of the analysis. The methods can be generalized to other double-diffusive convection problems.

## 1. Introduction

A layer with a stable temperature gradient and an unstable salt gradient, that is stably stratified in density, is known to be unstable to a double-diffusive instability, called salt fingering (Stern 1960). For a full list of references see the recent review of Turner (1985). It has long been known that the aspect ratio of the convection is small (the cells are tall and thin) and this can be understood both physically (since is allows the system to act as an efficient heat exchanger) and by reference to the governing equations, where the most rapidly growing mode turns out to have a well-defined small horizontal scale (see, e.g. Holyer 1981 and §2). Experimental observation of the salt-finger planform (Shirtcliffe & Turner 1970) shows that the cells take the form of irregular squares (rather than two-dimensional 'sheets'), with each upward-moving region surrounded by four downward-moving ones. Theoretical justification for such a structure is notably absent from the literature. Any researcher 'knows', when asked, that square cells act as more efficient heat exchangers than do sheets, but all theories of the finger regime depend on a 'mean-field' approach (see for example the theory given on p. 282 of Turner 1973), where the equations involve horizontal derivatives only as the horizontal Laplacian $\nabla_H^2 \equiv (\partial^2/\partial x^2)+(\partial^2/\partial y^2)$, so that all horizontal dependences $f(x, y)$ are equivalent provided $\nabla_H^2 f = -\alpha^2 f$ for any fixed $\alpha$. Any attempt to understand the planform must involve getting away from mean-field theories and considering the nonlinear terms that actually determine the planform. The only attempt to address the problem directly was made by Straus (1972) who calculated numerically two-dimensional finite-amplitude solutions to our equations (2.1)–(2.4) and then investigated their stability to three-dimensional disturbances. He found that the two-dimensional solutions were always stable in the finger regime. However his analysis does not completely resolve the problem as he restricted his work to the limits $\tau \equiv \kappa_S/\kappa_T \rightarrow 0$, $\sigma \equiv \nu/\kappa_T \rightarrow \infty$, where $\kappa_S$, $\kappa_T$, $\nu$ are the diffusivities of salt and heat, and kinematic viscosity respectively. Though for water $\tau$ is approximately $\frac{1}{80}$ its Prandtl number $\sigma$ is about 7 which is not very large, besides which salt fingers can be found in systems where $\sigma$ is not large and $\tau$ is not small. Also, since the stability of three-dimensional solutions was not investigated, the possibility of a bistable solution cannot be ruled out. For these reasons it seemed worthwhile to investigate less extreme parameter ratios, using a two-scale approach that allows a direct determination of the stability of both two- and three-dimensional planforms. The expansion scheme leads to nonlinear evolution equations for the vertical structure of the convection: the effect of the horizontal dependence is 'integrated out'. This works because the nonlinear terms in the equation vanish indentically for infinitely long 'cells'. The approach is almost identical to that used by Normand (1984) to discuss Bénard convection in tall containers. She did not address the planform selection problem, and indeed our leading-order equations, while they rule out convection with hexagonal planforms, or any other form in which resonant triad interactions are present, are degenerate as between squares and sheets. Carrying the analysis to higher order does eventually lead to an equation describing planform selection. Unfortunately it shows once again that sheets are the preferred form of motion in parameter ranges where the analysis is valid. While this is consistent with the conclusions of Straus (1972) and Swift (1984) who showed that thermohaline convection takes the form of sheets at onset (when the aspect ratio is of order unity), it clearly does not describe correctly the parameter range appropriate to laboratory salt fingers, which has higher Rayleigh and Péclet numbers than the analysis can handle. Presumably the observed square-cell structure is the result of an instability of 'cross-roll' type that only sets in when there are relatively thin boundary layers at the extremities of the convection cells. Although the results are thus of rather a negative kind, they do point the way to the appropriate (and very difficult) problem to be tackled if all the relevant physics is to be included. They also show that, as for convection with *large* aspect ratio (Chapman & Proctor 1980) it is possible to go beyond the usual weakly nonlinear stability theory and derive results that are valid over much wider parameter ranges.

The plan of the paper is as follows: in §2 we discuss the linearized stability theory, while in §3 we give the nonlinear analysis which leads to the determination of the planform. In a Conclusion we discuss the shortcomings of the theory, analyse the effect of different boundary conditions and indicate possible generalizations.

## 2. Equations and linearized stability problem

### 2.1. *Dimensionless equations*

For our model we take a fluid layer of depth $d$ with horizontal stress-free boundaries at $z = 0, d$. At $z = 0$ the temperature and salinity are $T_0$, $\Sigma_0$ while at $z = d$ they are $T_0 + \Delta T$, $\Sigma_0 + \Delta S$ respectively. Making the usual non-dimensionalization, the dimensionless equations describing the convection are

$$\frac{1}{\sigma}\left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right] = -\nabla p - R_T\,\theta\hat{\boldsymbol{z}} + R_S\,S\hat{\boldsymbol{z}} + \nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \boldsymbol{u}\cdot\hat{\boldsymbol{z}} + \nabla^2\theta, \tag{2.2}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{u}\cdot\nabla S = \boldsymbol{u}\cdot\hat{\boldsymbol{z}} + \tau\nabla^2 S, \tag{2.3}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2.4}$$

where, if $\nu$ is viscosity, $\kappa_T$ the thermal diffusivity and $\kappa_S$ salt diffusivity, the velocity $\boldsymbol{u}$ is scaled with $\kappa_T/d$, time $t$ with $d^2/\kappa_T$ and pressure with $(\nu\kappa_T/d^2)$. The temperature perturbation $\theta$ and salt perturbation $S$ are defined by

$$\left.\begin{aligned} T &= T_0+\Delta T\left[\frac{z}{d}-\theta\right],\\ \Sigma &= \Sigma_0+\Delta S\left[\frac{z}{d}-S\right],\end{aligned}\right\} \tag{2.5}$$

and the dimensionless parameters are

$$\left.\begin{aligned} \sigma &= \frac{\nu}{\kappa_T}, \quad R_T = \frac{g\tilde{\alpha}\Delta T d^3}{\kappa_T\nu},\\ \tau &= \frac{\kappa_S}{\kappa_T}, \quad R_S = \frac{g\tilde{\beta}\Delta S d^3}{\kappa_T\nu},\end{aligned}\right\} \tag{2.6}$$

where $g$ is the gravitational acceleration and $-\tilde{\alpha}$, $\tilde{\beta}$ are respectively the coefficients of density variation with respect to temperature and salinity. In the salt finger regime both $R_T$ and $R_S$ are positive. For salt fingers $\tau < 1$. The boundary conditions at $z = 0, 1$ (in dimensionless units) are, for the present,

$$\theta = S = \boldsymbol{u}\cdot\hat{\boldsymbol{z}} = \frac{\partial}{\partial z}(\boldsymbol{u}\times\hat{\boldsymbol{z}}) = 0. \tag{2.7}$$

(We shall discuss other boundary conditions later.)

The linearized version of these equations has been extensively studied (Baines & Gill 1969; Huppert & Moore 1976; Da Costa, Knobloch & Weiss 1981; Knobloch & Proctor 1981; Schmitt 1983) and we do not propose to go into details here. We recall only that instability always sets in as *steady* convection if $\tau < 1$, and occurs at

$$\frac{R_S}{\tau} = R_T+\frac{(\pi^2+\alpha^2)^3}{\alpha^2}, \tag{2.8}$$

where $\theta \propto f(x,y)\sin\pi z$, $\nabla_H^2 f \equiv (\partial^2 f/\partial x^2)+(\partial^2 f/\partial y^2) = -\alpha^2 f$. Thus near marginal stability the growing modes all have values of $\alpha$ close to $\pi/\sqrt{2}$ (giving a minimum of $R_T+\frac{27}{4}\pi^4$ for the right-hand side of (2.8)). However, if the convection is highly supercritical it turns out that the mode of maximum exponential growth rate has $\alpha \gg 1$.

If we linearize the equations and express the equation of motion in terms of $w \equiv \boldsymbol{u}\cdot\hat{\boldsymbol{z}}$ alone by eliminating the pressure, then if the exponential growth rate is $\lambda$ we have

$$\frac{\lambda}{\sigma}(-\nabla^2 w) = R_T\nabla_H^2\theta - R_S\nabla_H^2 S - \nabla^4 w, \tag{2.9a}$$

$$\lambda\theta = w+\nabla^2\theta, \tag{2.9b}$$

$$\lambda S = w+\tau\nabla^2 S, \tag{2.9c}$$

and if $\theta \propto f(x,y)\sin\pi z$ as before, we get

$$\frac{\lambda}{\sigma}(\pi^2+\alpha^2)w = -R_T\alpha^2\theta+R_S\alpha^2 S-(\pi^2+\alpha^2)^2 w, \tag{2.10a}$$

$$\lambda\theta = w-(\pi^2+\alpha^2)\theta, \tag{2.10b}$$

$$\lambda S = w-\tau(\pi^2+\alpha^2)S. \tag{2.10c}$$

Then we can find a cubic equation for $\lambda$, whose asymptotic behaviour for small $\tau$, or large $R_S$ can be calculated. As an example consider the case

$$\tau = O(1), \quad R_S = O(R_T) \gg 1.$$

On the assumption that $\alpha \gg 1$, the growth rate $\lambda$ satisfies the equation

$$\left(\frac{\lambda}{\sigma}+\alpha^2\right)(\lambda+\alpha^2)(\lambda+\tau\alpha^2) = -R_T(\lambda+\tau\alpha^2)+R_S(\lambda+\alpha^2), \tag{2.11}$$

and if $\lambda = R_T^{\frac{1}{2}}\tilde{\lambda}$, $\alpha = R_T^{\frac{1}{4}}\tilde{\alpha}$, $\gamma = R_S/R_T$ then

$$\left(\frac{\tilde{\lambda}}{\sigma}+\tilde{\alpha}^2\right)(\tilde{\lambda}+\tilde{\alpha}^2)(\tilde{\lambda}+\tau\tilde{\alpha}^2) = -(\tilde{\lambda}+\tau\tilde{\alpha}^2)+\gamma(\tilde{\lambda}+\tilde{\alpha}^2), \tag{2.12}$$

and it may be verified that the maximum of $\tilde{\lambda}$ occurs when $\tilde{\alpha} = O(1)$, if $\gamma$ is not too large.

Other limits can be constructed (see for example Stern 1960), but they all have the same feature: that convection can be expected to occur as narrow cells. Indeed observations of salt fingers (e.g. Shirtcliffe & Turner 1970) indicate that 'finger' interfaces consist of thin fluid columns, on a scale quite close to that given by the maximum growth mode.

However, these models take no account of the finite vertical extent of the layer, since all vertical derivatives are neglected. If there is no vertical dependence, the solutions given above actually solve the full nonlinear equations (2.1)–(2.4) since the nonlinear terms vanish identically. For a finite layer the nonlinear terms will be important, if only in relatively thin boundary layers at the extremities. Furthermore, the $z$-independent models have nothing to say about the *planform* of the motion since any function $f(x,y)$ satisfying the membrane equation $\nabla^2 f = -\alpha^2 f$ is equally possible. The observed planform of salt fingers appears to approximate a 'square-cell' tessellation, with rising-fluid regions surrounded by four sinking regions (see figure 8.18 of Turner 1973). This selection must be due to nonlinear interactions between different modes satisfying the same membrane equation.

The analysis that follows exploits the disparity of horizontal and vertical lengthscales to include the nonlinear interaction terms. The methods used are similar to those of Normand (1984) who considers Bénard convection in tall containers (but does not address the planform selection problem) and Soward (1974) who considers convection in a rapidly rotating system. The details of the latter calculation are quite different, however, due to the different nature of the physical constraints. The following section consists of the development of an expansion procedure with (essentially) the aspect ratio of the cells as the small parameter. Evolution equations are found for the vertical structure of the convection and although the planform selection problem remains degenerate at leading order it can be resolved by proceeding further with the expansion.

### 3.1. *The expansion scheme*

For the nonlinear calculations it is convenient to adopt the *horizontal* (rather than vertical) scale as unit of length. In fact, instead of $d$, we use

$$l \equiv \left[\frac{\kappa_T \nu}{g\alpha\Delta T}\right]^{\frac{1}{3}}, \tag{3.1}$$

so that in this scaling $R_T$ is identically unity. We suppose that in the new scaling $R_S = O(1)$ and that $\tau < 1$ but $\tau$ is not necessarily very small. Then equations

(2.1)–(2.4) still hold but with $R_T = 1$. We now suppose that the dimensionless layer depth is $\tilde{d}$, where

$$\tilde{d} \equiv \frac{d}{l} = \epsilon^{-1} \gg 1, \tag{3.2}$$

and make the scalings

$$\left.\begin{aligned} &\frac{\partial}{\partial t} = \epsilon^3 \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial z} = \epsilon \frac{\partial}{\partial \xi}, \quad \boldsymbol{u} = (\epsilon^2 \tilde{\boldsymbol{v}}, \epsilon \tilde{w}) \quad \left[\text{so } \boldsymbol{\nabla}_{\mathrm{H}} \cdot \tilde{\boldsymbol{v}} + \frac{\partial \tilde{w}}{\partial \xi} = 0\right], \\ &\theta = \epsilon \Theta(\xi, T) + \epsilon \tilde{\theta}, \quad S = \epsilon \Sigma(\xi, T) + \epsilon \tilde{S}, \quad p = \epsilon^2 P(\xi, T) + \epsilon^2 \tilde{p}, \end{aligned}\right\} \tag{3.3}$$

(where $\bar{S} = \epsilon \Sigma$, etc. and the overbar denotes horizontal average; $\bar{\tilde{\boldsymbol{v}}} = \bar{\tilde{w}} = 0$). Note that $w$ is $O(1)$ when scaled with $\kappa_T/d$, whereas in the usual perturbation theories $w = o(\kappa_T/d)$. Then substituting into the equations (2.1)–(2.4) and dropping the tildes we obtain

$$\frac{\epsilon^2}{\sigma}\left[\frac{\partial \boldsymbol{v}}{\partial T} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\mathrm{H}}) \boldsymbol{v} + w \frac{\partial \boldsymbol{v}}{\partial \xi}\right] = -\boldsymbol{\nabla}_{\mathrm{H}} p + \left[\nabla_{\mathrm{H}}^2 + \epsilon^2 \frac{\partial^2}{\partial \xi^2}\right] \boldsymbol{v}, \tag{3.4}$$

$$\frac{\epsilon^2}{\sigma}\left[\frac{\partial w}{\partial T} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\mathrm{H}}) w + w \frac{\partial w}{\partial \xi} - \frac{\partial}{\partial \xi}(\overline{w^2})\right] = -\epsilon^2 \frac{\partial p}{\partial \xi} + \left[\nabla_{\mathrm{H}}^2 + \epsilon^2 \frac{\partial^2}{\partial \xi^2}\right] w - \theta + R_S S, \tag{3.5}$$

$$\epsilon^2 \left[\frac{\partial \theta}{\partial T} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\mathrm{H}}) \theta + w \frac{\partial \theta}{\partial \xi} + w \frac{\partial \Theta}{\partial \xi} - \frac{\partial}{\partial \xi}(\overline{w\theta})\right] = w + \left[\nabla_{\mathrm{H}}^2 + \epsilon^2 \frac{\partial^2}{\partial \xi^2}\right] \theta, \tag{3.6}$$

$$\epsilon^2 \left[\frac{\partial S}{\partial T} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}_{\mathrm{H}}) S + w \frac{\partial S}{\partial \xi} + w \frac{\partial \Sigma}{\partial \xi} - \frac{\partial}{\partial \xi}(\overline{wS})\right] = w + \tau \left[\nabla_{\mathrm{H}}^2 + \epsilon^2 \frac{\partial^2}{\partial \xi^2}\right] S, \tag{3.7}$$

$$\frac{\partial \Theta}{\partial T} + \frac{\partial}{\partial \xi}(\overline{w\theta}) = \frac{\partial^2 \Theta}{\partial \xi^2}, \tag{3.8}$$

$$\frac{\partial \Sigma}{\partial T} + \frac{\partial}{\partial \xi}(\overline{wS}) = \tau \frac{\partial^2 \Sigma}{\partial \xi^2}, \tag{3.9}$$

while $P(\xi, T)$ is determined from the mean of the vertical momentum equation. We solve these equations by expanding all variables, including $R_S$, in powers of $\epsilon$. We write

$$w = w_0 + \epsilon^2 w_1 + \dots \text{ etc}; \quad R_S = R_{S0} + \epsilon^2 R_{S1} + \dots, \tag{3.10}$$

and then at leading order we obtain

$$\left.\begin{aligned} 0 &= \nabla_{\mathrm{H}}^2 w_0 - \theta_0 + R_{S0} S_0, \\ 0 &= w_0 + \nabla_{\mathrm{H}}^2 \theta_0, \\ 0 &= w_0 + \tau \nabla_{\mathrm{H}}^2 S_0; \end{aligned}\right\} \tag{3.11}$$

this is of course the linear stability problem considered in §2. Equations (3.11) are solved by

$$\left.\begin{aligned} &w_0 = \alpha^2 \theta_0 = \tau \alpha^2 S_0; \quad R_{S0} = \tau(1 + \alpha^4); \\ &w_0 = \sum_n f^{(n)}(x, y) A^{(n)}(\xi, T) \end{aligned}\right\} \tag{3.12}$$

where $\nabla_{\mathrm{H}}^2 f^{(n)} = -\alpha^2 f^{(n)}$, and the $f^{(n)}$ and the $A^{(n)}$ are otherwise arbitrary at this stage, apart from boundary conditions on $A^{(n)}$ which we discuss later. It may be noted that (3.12) does not yield a minimum of $R_{S0}$ as a function of $\alpha$. This is because, in order to effect an expansion scheme, we are forced to choose $R_S$ close to its value on the marginal stability curve. An attempt to choose $R_S$ large enough so as to capture the mode of maximum growth rate leads to an intractable boundary-layer problem. Nonetheless, the expansion scheme used here allows us to go further from the marginal curve ($R_S - R_{S0} = O(\epsilon\tau)$) than do conventional methods, for which $R_S - R_{S0} \approx O(\epsilon^2\tau^2)$. The analysis of §2 and the experiments justify the consideration of long thin cells and our analysis makes no attempt to treat the relative stability of different wavenumbers $\alpha$.

If we write $\Theta_0 = B(\xi, T)$, $\Sigma_0 = C(\xi, T)$ then these variables satisfy the equations

$$B_T - B_{\xi\xi} = \left[-\frac{1}{\alpha^2}\sum_m \sum_n \overline{w^{(m)} w^{(n)}}\right]_\xi, \tag{3.13a}$$

$$C_T - \tau C_{\xi\xi} = \left[-\frac{1}{\tau\alpha^2}\sum_m \sum_n \overline{w^{(m)} w^{(n)}}\right]_\xi, \tag{3.13b}$$

where $w^{(m)} = f^{(m)}A^{(m)}$, etc. Now every solution of the membrane equation for $f^{(m)}$ can be written in the form

$$f^{(m)} = e^{i\boldsymbol{a}^{(m)}\cdot\boldsymbol{x}}, \quad |\boldsymbol{a}^{(m)}|^2 = \alpha^2, \quad \boldsymbol{a}^m \cdot \hat{\boldsymbol{z}} = 0, \tag{3.14}$$

so if $\boldsymbol{a}^{(-m)} = -\boldsymbol{a}^{(m)}$, $A^{(-m)} = A^{(m)*}$ (to ensure that $w_0$ is real) then clearly

$$\overline{f^{(m)}f^{(n)}} = \delta_{m,-n}$$

and so

$$\sum_m \sum_n \overline{w^{(m)}w^{(n)}} = \sum_m |A^{(m)}|^2. \tag{3.15}$$

At higher order in $\epsilon^2$ we obtain a sequence of inhomogeneous problems of the form

$$P_n = \nabla_{\mathrm{H}}^2 w_n - \theta_n + R_{S0} S_n, \tag{3.16a}$$

$$Q_n = \nabla_{\mathrm{H}}^2 \theta_n + w_n, \tag{3.16b}$$

$$R_n = \tau\nabla_{\mathrm{H}}^2 S_n + w_n, \tag{3.16c}$$

where $P_n$, $Q_n$, $R_n$ can be evaluated in terms of known quantities. The Fredholm Alternative for these equations can be written in the form

$$\alpha^2\overline{P_n f^{(j)}} - \overline{Q_n f^{(j)}} + (1+\alpha^4)\,\overline{R_n f^{(j)}} = 0, \tag{3.17}$$

for every $f^{(j)}$. When $w_n$ has been found the horizontal velocity components and the pressure may be found by substitution in (3.4), 13.5).

Thus for $n = 1$ we substitute into (3.4)–(3.9) and obtain, after some algebra, the following set of equations:

$$0 = F\frac{\partial A^{(i)}}{\partial T} + E\sum_j \sum_k \left[\frac{\partial A^{(j)}}{\partial \xi} A^{(k)} G(i;j,k)\right] - 3\alpha^2\frac{\partial^2 A^{(i)}}{\partial \xi^2} - \frac{R_{S1}}{\tau}A^{(i)} + A^{(i)}\left[(1+\alpha^4)\frac{\partial C}{\partial \xi} - \frac{\partial B}{\partial \xi}\right], \tag{3.18}$$

where

$$F = \frac{\sigma(1-\tau)+(\sigma+\tau)\alpha^4}{\sigma\tau\alpha^2} > 0, \tag{3.19a}$$

$$E = \frac{3}{2}\left[\frac{\alpha^2}{\sigma} + \frac{\alpha^2}{\tau} + \frac{1}{\tau\alpha^2}(1-\tau)\right] > 0, \tag{3.19b}$$

and $G(i;j,k)$ is zero unless $\boldsymbol{a}^{(i)} = \boldsymbol{a}^{(j)} + \boldsymbol{a}^{(k)}$, when $G = 1$. That is, the function $G$ picks out resonant triads among the membrane functions $f^{(j)}$. Equation (3.18) differs from the usual set of evolution equations for instabilities with aspect ratios of order unity in that it contains terms *quadratic* in the amplitudes $A^{(m)}$ (cf. Normand 1984). It must be solved in conjunction with the mean equations (3.13) and so is not simply reducible to one of Landau–Ginzberg form.

To effect a solution to (3.13), (3.18) boundary conditions on $A^{(i)}$, $B$ and $C$ must be prescribed. For the simplest case of stress-free, fixed temperature and salinity boundaries, we have (cf. (2.7))

$$w = \theta = S = \frac{\partial^2 w}{\partial z^2} = \Theta = \Sigma = 0 \quad \text{at } z = 0, \epsilon^{-1}, \tag{3.20}$$

and we see that all these conditions can be satisfied by setting

$$A^{(j)} = B = C = 0, \quad \xi = 0, 1. \tag{3.21}$$

For other types of boundary condition we cannot satisfy all the conditions (3.20) at leading order and boundary layers arise at $\xi = 0, 1$. We discuss these in the Appendix, but in fact they make no difference to the solution at leading order.

### 3.2. *Hexagons and squares*

We first examine a disturbance in which only three modes are present and form a resonant triad. This leads to three equations of the form

$$0 = F\frac{\partial A^{(1)}}{\partial T} + E\frac{\partial}{\partial \xi}[A^{(2)}A^{(3)}] - 3\alpha^2\frac{\partial^2 A^{(1)}}{\partial \xi^2} - \frac{R_{S1}}{\tau}A^{(1)} + A^{(1)}\left\{(1+\alpha^4)\frac{\partial C}{\partial \xi} - \frac{\partial B}{\partial \xi}\right\} \tag{3.22}$$

(where others are obtained by cyclic permutation of indices) which have been solved numerically together with (3.13) for a variety of values of $\sigma$, $\tau$, $\alpha$ and $R_{S1}$. In every case the solution tended to a steady state with only one of the $A^{(i)}$'s non-zero (a 'sheet' solution) except when the three amplitudes were exactly equal initially. Thus it seems that the system acts to remove any resonant triad interactions. This result is in keeping with intuition in that hexagonal convection seems invariably to be associated with a lack of up–down symmetry in the underlying physics.

Guided by the above considerations, we now focus on disturbances which contain no resonant triad interactions – and the simplest of these is the case

$$w_0 = A^{(1)}\cos\alpha x + A^{(2)}\cos\alpha y, \tag{3.23}$$

(note the slight change of notation) which corresponds to a 'square cell' (cf. Jenkins & Proctor 1984) if $A^{(1)} = A^{(2)}$. Thus the choice (3.23) will enable us to decide whether rolls or squares are preferred in this parameter range. The equations now reduce to

$$FA^{(1)}_T = 3\alpha^2 A^{(1)}_{\xi\xi} + \mu A^{(1)} - A^{(1)}[(1+\alpha^4)C_\xi - B_\xi] \tag{3.24a}$$

$$FA^{(2)}_T = 3\alpha A^{(2)}_{\xi\xi} + \mu A^{(2)} - A^{(2)}[(1+\alpha^4)C_\xi - B_\xi], \tag{3.24b}$$

$$B_T = B_{\xi\xi} - \frac{1}{2\alpha^2}[(A^{(1)})^2 + (A^{(2)})^2]_\xi, \tag{3.24c}$$

$$C_T = \tau C_{\xi\xi} - \frac{1}{2\tau\alpha^2}[(A^{(1)})^2 + (A^{(2)2}]_\xi, \tag{3.24d}$$

where $\mu \equiv R_{S1}/\tau$. To investigate this set of equations it is convenient to write

$$A^{(1)} + \mathrm{i}A^{(2)} = \rho(\xi, T)\,\mathrm{e}^{\mathrm{i}\phi(\xi, T)} \tag{3.25}$$

and then we obtain

$$F\rho_T = 3\alpha^2[\rho_{\xi\xi}-\rho\phi_\xi^2]+\mu\rho-\rho[(1+\alpha^4)C_\xi-B_\xi], \tag{3.26a}$$

$$F\rho\phi_T = \frac{3\alpha^2}{\rho}[\rho^2\phi_\xi]_\xi, \tag{3.26b}$$

$$B_T = B_{\xi\xi}-\frac{1}{2\alpha^2}(\rho^2)_\xi, \tag{3.26c}$$

$$C_T = \tau C_{\xi\xi}-\frac{1}{2\tau\alpha^2}(\rho^2)_\xi. \tag{3.26d}$$

It is thus clear that the system is totally degenerate at this order: for any 'phase' $\phi$ constant in $\xi$ and $T$, there is a steady solution of the same 'amplitude' $\rho$ and since the value of $\phi$ itself occurs nowhere in the equations, there is no way to determine a preferred value of it. The 'diffusion-equation'-like form of (3.26*b*) suggests that $\phi$ will in fact tend to a constant value everywhere as $T\to\infty$, and although there is no proof available, numerical simulations of (3.26) show that this is what occurs. In order to resolve the degeneracy, we need to proceed to higher order in the expansion scheme, but before doing this it is instructive to examine the steady solution to (3.26) so as to determine the limits of the scheme.

### 3.3. *Steady solutions*

If we set $\partial/\partial T = 0$ in (3.26*c*, *d*) these equations may be integrated to give

$$C_\xi = \frac{1}{\tau^2}B_\xi = \frac{1}{2\alpha^2}(\rho^2-\langle\rho^2\rangle); \quad \langle\rho^2\rangle = \int_0^1 \rho^2\,\mathrm{d}\xi. \tag{3.27}$$

Thus both terms in the square brackets in (3.26*a*) have the same form, and if factors of order unity are removed by appropriate scaling of $\mu$ and the amplitude of $\rho$ we arrive at the canonical equation

$$0 = \rho_{\xi\xi}+\mu\rho-(\rho^2-\langle\rho^2\rangle)\rho. \tag{3.28}$$

This equation, with the boundary conditions $\rho(0) = \rho(1) = 0$, can actually be solved for any $\mu > \pi^2$ in terms of Jacobian elliptic functions. (For $\mu < \pi^2$ there is no steady convection as we are then below the marginal stability boundary.) In fact, if $E(m)$ and $K(m)$ are the usual elliptic integrals of the first and second kinds respectively, then the solution is given parametrically by the relations

$$N = \mu+\langle\rho^2\rangle, \tag{3.29a}$$

$$K(m) = \frac{1}{2}\left[\frac{N}{1+m}\right]^{\frac{1}{2}}, \tag{3.29b}$$

$$\langle\rho^2\rangle = \frac{8K(m)}{m}[K(m)-E(m)], \tag{3.29c}$$

$$\rho = \left[\frac{2mN}{1+m}\right]^{\frac{1}{2}}\mathrm{sn}\left[\left(\frac{N}{1+m}\right)^{\frac{1}{2}}\xi\,|\,m\right], \tag{3.29d}$$

(cf. Knobloch & Proctor 1981). When $\mu \gg 1$, the parameter $m$ is very close to 1 and $\mathrm{sn}\,(x\,|\,m)$ is then approximately constant away from the boundaries. We can show using boundary-layer techniques that for large $\mu$, $\rho \approx \mu/2\sqrt{2}$ in the interior. Thus the amplitude of the convection increases like $(R_S-R_{S0})$, and when $\mu$ is large the associated boundary layer thickness is $O(\mu^{-1})$. Hence the separation of scales on which the derivation of (3.24) was based can be expected to hold until $\mu \approx \epsilon^{-1}$, or $R_S - R_{S_0} = O(\epsilon)$. Even this is somewhat too small accurately to reflect the situation in most experiments. An attempt to investigate the stability problem in the latter parameter range leads to a difficult nonlinear boundary-layer analysis near $\xi = 0, 1$ and this has not yet been solved.

### 3.4. *Resolution of the degeneracy*

In order to distinguish between rolls and squares, we need to proceed to the next order in the expansion scheme. Bearing in mind the number of terms contained in $P_2$, $Q_2$, $R_2$ this represents a formidable task. However, we can save time by noting that of the higher-order corrections to (3.24) we are only interested in those terms that give information about the absolute phase. In fact all the $w_1$, $\theta_1$, $S_1$ terms are of five types, namely those proportional to $\cos\alpha x$, $\cos\alpha y$, $\cos 2\alpha x$, $\cos 2\alpha y$, $\cos\alpha x \cos\alpha y$. The first two give no information about the phase and the third and fourth turn out to be identically zero since the temperature, salinity and vertical velocity fields all turn out to be proportional to each other at leading order in the steady state. Lengthy calculation yields, writing $w_1 = C_w \cos\alpha x \cos\alpha y (A^{(1)}A^{(2)})_\xi +$ (other terms), etc.,

$$C_w = -\frac{1}{3\alpha^8\tau}\left[\alpha^6\left(1+\frac{2\tau}{\sigma}\right)+\alpha^2(1-\tau)\right], \tag{3.30a}$$

$$C_\theta = -\frac{1}{6\alpha^8\tau}\left[2\alpha^4\tau\left(2+\frac{1}{\sigma}\right)+(1-\tau)(1+\alpha^2)\right], \tag{3.30b}$$

$$C_S = -\frac{1}{6\alpha^8\tau}\left[2\alpha^4\left(2+\frac{1}{\sigma}\right)+\left(\frac{1}{\tau}-1\right)(1+4\alpha^4)\right], \tag{3.30c}$$

and it can be seen that all these are negative when $\tau < 1$. Having (formally) solved for $w_1$ etc. we may now write down the solvability condition (3.17) at $O(\epsilon^4)$. However, it is better to combine the $O(\epsilon^2)$ and $O(\epsilon^4)$ results to yield the mixed-order system (recall equation (3.25))

$$\begin{aligned} &FA^{(1)}_T - 3\alpha^2 A^{(1)}_{\xi\xi} - \mu A^{(1)} + A^{(1)}[(1+\alpha^4)C_\xi - B_\xi] \\ &\quad + \tfrac{1}{2}\epsilon^2\left[A^{(2)}(A^{(1)}A^{(2)})_{\xi\xi}\left\{\tfrac{1}{4}FC_w + \frac{\alpha^2}{2\sigma}C_w - \tfrac{1}{2}C_\theta + \tfrac{1}{2}(1+\alpha^4)C_S\right\}\right. \\ &\quad \left. + A^{(2)}_\xi(A^{(1)}A^{(2)})_\xi\left\{\tfrac{1}{2}FC_w + \frac{\alpha^2}{2\sigma}C_w - \tfrac{1}{2}C_\theta + \tfrac{1}{2}(1+\alpha^4)C_S\right\}\right] \\ &\qquad = \epsilon^2[\text{cubic terms proportional to } (A^{(1)}A^{(2)}_{\xi\xi} - A^{(2)}A^{(1)}_{\xi\xi})] \\ &\qquad\quad + \epsilon^2[\text{terms proportional to } \cos\phi], \end{aligned} \tag{3.31}$$

and similarly for $A^{(2)}$,

$$B_T - B_{\xi\xi} + \frac{1}{2\alpha^2}(A^{(1)^2} + A^{(2)^2})_\xi = \epsilon^2[\text{terms independent of } \phi]$$

$$C_T - \tau C_{\xi\xi} + \frac{1}{2\tau\alpha^2}(A^{(1)^2} + A^{(2)^2})_\xi = \epsilon^2[\text{terms independent of } \phi].$$

Now at leading order in $\epsilon$, as we have seen, $A^{(1)} \to \rho\cos\phi$, $A^{(2)} \to \rho\sin\phi$ where $\phi$ is an unknown constant. When the higher-order terms are included it is clear that $\phi$

will itself evolve on the slow timescale $T^* = \epsilon^2 T$. For satisfaction of the equations at $O(\epsilon^0)$ $A^{(1)}$ and $A^{(2)}$ must be of the form

$$W \equiv A^{(1)} + \mathrm{i}A^{(2)} = \rho \mathrm{e}^{\mathrm{i}(\hat{\phi}(T^*)+\epsilon^2\phi_2(\xi,\, T^*))} + \epsilon^2 D\rho \mathrm{e}^{\mathrm{i}\hat{\phi}T^*} + o(\epsilon^2), \tag{3.32}$$

where $D$ is real and $\rho(\xi)$ may be taken to satisfy the steady amplitude equation (3.28). Substituting the *ansatz* (3.32) into (3.31) then leads to the cancellation of all terms at leading order. At order $\epsilon^2$ we obtain

$$F\frac{\partial W}{\partial T^*} + \tfrac{1}{2}(\cos\hat{\phi}\,\sin^2\hat{\phi} + \mathrm{i}\,\sin\hat{\phi}\,\cos^2\hat{\phi})\,[P\rho(\rho^2)_{\xi\xi} + Q\rho_\xi(\rho^2)_\xi]$$
$$-3\alpha^2\left[\frac{\mathrm{i}}{\rho}\,(\rho^2\phi_{2\xi})_\xi\right]\mathrm{e}^{\mathrm{i}\hat{\phi}} = (\text{terms proportional to } \mathrm{e}^{\mathrm{i}\hat{\phi}}) + O(\epsilon^2), \tag{3.33}$$

where $P = \frac{1}{4}FC_W + (\alpha^2/2\sigma)\,C_w - \frac{1}{2}C_\theta + \frac{1}{2}(1+\alpha^4)\,C_S$, $Q = P + \frac{1}{4}FC_w$, and the last term on the left-hand side derives from the 'out of phase' part of $(-3\alpha^2 W_{\xi\xi})$ (cf. (3.18), etc.). Multiplying (3.33) by $\mathrm{e}^{-\mathrm{i}\hat{\phi}}$, taking the imaginary part and ignoring all small terms we obtain

$$F\rho\frac{\partial\hat{\phi}}{\partial T^*} + \tfrac{1}{2}(\sin\hat{\phi}\,\cos^3\hat{\phi} - \cos\hat{\phi}\,\sin^3\hat{\phi})\,(P\rho(\rho^2)_{\xi\xi} + Q\rho_\xi(\rho^2)_\xi) - \frac{3\alpha^2}{\rho}\,(\rho^2\phi_{2\xi})_\xi = 0. \tag{3.34}$$

Now this equation still contains the unknown function $\phi_2$. However, for solutions that are not singular at the boundaries, we must have

$$\int_0^1 (\rho^2\phi_{2\xi})_\xi\,\mathrm{d}\xi = [\rho^2\phi_{2\xi}]_0^1 = 0, \tag{3.35}$$

since $\rho$ vanishes at each boundary. Applying this to (3.34) we arrive at the required equation for $\hat{\phi}$, namely

$$\left(\int_0^1 \rho^2\,\mathrm{d}\xi\right) F\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}T^*} + \tfrac{1}{2}(\sin\hat{\phi}\,\cos^3\hat{\phi} - \cos\hat{\phi}\,\sin^3\hat{\phi})\left[P\int_0^1 \rho^2(\rho^2)_{\xi\xi}\,\mathrm{d}\xi + Q\int \rho\rho_\xi(\rho^2)_\xi\,\mathrm{d}\xi\right] = 0. \tag{3.36}$$

Since

$$\int_0^1 \rho^2(\rho^2)_{\xi\xi}\,\mathrm{d}\xi = -\int_0^1 [(\rho^2)_\xi]^2\,\mathrm{d}\xi, \tag{3.37a}$$

$$\int_0^1 (\rho\rho_\xi)\,(\rho^2)_\xi\,\mathrm{d}\xi = \frac{1}{2}\int_0^1 [(\rho^2)_\xi]^2\,\mathrm{d}\xi, \tag{3.37b}$$

we finally obtain

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}T^*} + \tfrac{1}{8}G\sin 4\hat{\phi} = 0, \tag{3.38}$$

where

$$G = (\tfrac{1}{2}Q - P)\,\frac{\int_0^1 [(\rho^2)_\xi]^2\,\mathrm{d}\xi}{F\int_0^1 \rho^2\,\mathrm{d}\xi}. \tag{3.39}$$

It may in fact be verified that $G > 0$ when $\tau < 1$ for all $\alpha$ and $\sigma$. The fixed points of (3.38) are $\hat{\phi} = \frac{1}{4}n\pi$, $n = 1, 2, \ldots$ and it is easy to see that the values $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, etc. (corresponding to square cells) are unstable, while 0, $\frac{1}{2}\pi$ etc. (sheet solutions) are stable. This conclusion, which is independent of the amplitude of the solution, agrees with the results of Straus (1972) and Swift (1984) for aspect ratios of order unity and weakly nonlinear convection.

## 4. Discussion

We have shown that thermohaline convection at small aspect ratio shows a preference for roll rather than square-cell tessellation for all values of the parameters such that thin salinity or temperature boundary layers do not dominate the dynamics. This result conflicts, of course, with observations of fingering interfaces that show the presence of more or less square cells. Two difficulties of the model might explain the discrepancy. First, the rather simple boundary conditions we have adopted may not be appropriate for an internal layer of fluid surrounded by well-mixed material. It may well be, for example, that the appropriate thermal boundary condition is one of fixed flux rather than fixed temperature. We have investigated the effect of changing the boundary conditions, and an outline of the results is given in the Appendix. A thin boundary layer of relative thickness $\epsilon$ forms at $\xi = 0, 1$ and this leads to $O(\epsilon)$ changes in the coefficients $P$, $Q$. Since the crucial quantity $G$ is bounded away from zero these small changes do not affect the above results. One might also wonder at the failure to predict the preferred wavenumber in the range considered. Unfortunately, any theory that has the (salt) Péclet number of order unity does not lend itself to the type of analysis pursued here, since boundary layers will develop as described in §3. A boundary-layer theory similar to that described by Howard (1965) may be undertaken if only the mean-field interactions are retained. This appears (though the details have not been fully worked out) to give a value for the 'preferred' wavenumber (i.e. that corresponding to the greatest heat transport) of the same order as that of the mode of maximum growth rate on linearized theory. However, any attempt to distinguish between different planforms by adding the 'out-of-phase' terms leads immediately to an intractable problem. There seems to be little doubt that there is a bifurcation of the 'cross-roll' type that occurs in the boundary-layer regime and leads to an (approximately) square-cell structure. (Some progress towards understanding the boundary layers has been made by Howard & Veronis (1984) but they have not as yet considered three-dimensional motion.)

Although the work presented here does not explain the experiments (which occur in a rather different parameter range) it does show that when convection occurs at small aspect ratio it is possible to extend the usual weakly nonlinear analysis (leading to 'normal forms' in the guise of low-order systems of o.d.e.'s) and construct systems of p.d.e.'s that allow the planform-selection question to be addressed in a larger parameter range than that possible with weakly nonlinear theories. Another type of double-diffusive convection exhibiting motion at small aspect ratio is magnetoconvection (see e.g. Proctor & Weiss 1982), and analogous methods may be used to find the planform in that case. Preliminary results are given in Proctor (1986).

## Appendix. The effect of varying the boundary conditions

The analysis of §3 was simple because the boundary conditions on $w$, $\theta$ and $S$ were all the same, and so could be satisfied by the single function $A$. If the temperature and salinity boundary conditions are changed this is no longer the case, and boundary layers appear at $\xi = 0, 1$. Since all the possible conditions lead to equivalent results, let us choose the case

$$w = \frac{\partial^2 w}{\partial \xi^2} = 0, \quad \frac{\partial S}{\partial \xi} = \frac{\partial \theta}{\partial \xi} = \frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \frac{\mathrm{d}\Sigma}{\mathrm{d}\xi} = 0, \quad \xi = 0, 1. \tag{A 1}$$

The leading-order solution (3.12) has $w_0 = \alpha^2\theta_0 = \tau\alpha^2 S_0$, which is incompatible with the above relations. If we now define a boundary-layer coordinate $z = \epsilon^{-1}\xi$ (near $\xi = 0$) and denote boundary-layer variables by $\tilde{\theta}$, etc. we set

$$\tilde{w} = \epsilon\tilde{w}_1, \quad \tilde{\theta} = \epsilon\tilde{\theta}_1, \quad \tilde{S} = \epsilon\tilde{S}_1, \tag{A 2}$$

and obtain the equation

$$0 = \alpha^2\tilde{\theta}_1 - \tau(1+\alpha^4)\,\alpha^2\tilde{S}_1 + (\partial_z^2 - \alpha^2)^2\,\tilde{w}_1, \tag{A 3a}$$

$$0 = \tilde{w}_1 + (\partial_z^2 - \alpha^2)\,\tilde{\theta}_1, \tag{A 3b}$$

$$0 = \tilde{w}_1 + \tau(\partial_z^2 - \alpha^2)\,\tilde{S}_1, \tag{A 3c}$$

which are to be solved subject to the conditions at $z = 0$;

$$\partial_z^2\tilde{w}_1 = 0, \quad \tilde{w}_1 = 0, \quad \frac{\partial\tilde{\theta}_1}{\partial z} = -\left.\frac{w_{0\xi}}{\alpha^2}\right|_{\xi=0}, \quad \frac{\partial\tilde{S}_1}{\partial z} = -\left.\frac{w_{0\xi}}{\tau\alpha^2}\right|_{\xi=0}. \tag{A 4}$$

$\tilde{w}_1$, $\tilde{\theta}_1$, $\tilde{S}_1$ all tend to constant values as $z \to \infty$.

The general solution of (A 3) satisfying the conditions as $z \to \infty$ is

$$\tilde{w}_1 = C + (\alpha^2 - m^2)\,D\mathrm{e}^{-mz} + (\alpha^2 - m^{*2})\,D^*\mathrm{e}^{-m^*z} \tag{A 5a}$$

$$\tilde{\theta}_1 = \frac{C}{\alpha^2} + D\mathrm{e}^{-mz} + D^*\mathrm{e}^{-m^*z}; \quad \tilde{S}_1 = \frac{\tilde{\theta}_1}{\tau}, \tag{A 5b}$$

where $C$, $D$, $D^*$ are constants, and $m$ and $m^*$ are the roots with positive real parts of the quartic equation

$$m^4 - 3\alpha^2 m^2 + 3\alpha^4 = 0. \tag{A 6}$$

The conditions at $z = 0$ then imply that

$$C + (\alpha^2 - m^2)\,D + (\alpha^2 - m^{*2})\,D^* = 0, \tag{A 7a}$$

$$m^2(\alpha^2 - m^2)\,D + m^{*2}(\alpha^2 - m^{*2})\,D^* = 0, \tag{A 7b}$$

$$mD + m^*D^* = \left.\frac{w_{0\xi}}{\alpha^2}\right|_{\xi=0}. \tag{A 7c}$$

Solving these equations yields the value of $C$ as

$$C = \frac{w_{0\xi}}{\alpha}\,(3 + 2\surd 3)^{-\frac{1}{2}}. \tag{A 8}$$

Thus the effect of the boundary layer is to replace the boundary condition $w_0 = 0$ on the velocity by the modification

$$w_0 = \frac{\epsilon w_{0\xi}}{\alpha}\,(3 + 2\surd 3)^{-\frac{1}{2}}; \quad \xi = 0 \tag{A 9}$$

with an analogous condition at $\xi = 1$. Different boundary conditions on $S$ and $\theta$ will give relations like (A 9) but with different numerical factors.

The small modification to the boundary conditions implied by (A 9) will clearly make changes of $O(\epsilon)$ in the leading-order dynamics. In particular the quantities $P$ and $Q$ will need to be adjusted by amounts of order $\epsilon$. Since these quantities differ by an amount of order unity, the sign of the quantity $G$ will not be affected.

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